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Combinational logic design

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CONTENTS

3.1 Introduction

In this lab, we will introduce simple methods to convert a logic expression into a *sum-of-products* expression. This will include performing necessary steps to reduce a sum-of-products expression to its simplest form, and to use Boolean algebra and the *Karnaugh map* as tools to simplify and design logic circuits. By the help of a truth table, you will be able to design simple logic circuits with it.

3.1.1 Sum-of-products Form

A sum-of-product expression consists of two or more `AND`($\cdot$, $\wedge$) terms that are `OR`ed($+$, $\vee$) together. For example:

$$F = ABC + \bar{A}B\bar{C} + A\bar{B}\bar{C}$$

$$F = AB + \bar{A}B\bar{C} + \bar{C}\bar{D} + D$$

Note that the *invert sign* usually cover one variable as it has different meaning, i.e., $\overline{AB} = \overline{(A \cdot B)}$ and $\bar{A}\bar{B} = \bar{A} \cdot \bar{B}$. The sum is the Boolean `OR` operation and the product is the Boolean `AND` operation.

3.1.2 Product-of-sums Form

The product-of-sums expression consists of two or more `OR` terms that are `AND`ed together. For example:

$$F = (A + \bar{B} + C)(A + C)$$

$$F = (A + \bar{B})(\bar{C} + D)F$$

3.1.3 Simplifying Logic Circuits

Many logic circuits are very complex at the first design, and it can be simplified. The objective is to reduce the logic circuit expression to a simpler form so that fewer gates, connections, and levels are required to construct the circuit. In Figure 3.1, the logic circuit represents the function

$$F = ABC + A\bar{B}\overline{(\bar{A}\bar{C})},$$

which can be simplified into the circuit represented in Figure 3.2. The equivalent Boolean function is

$$F = A(\bar{B} + C).$$

There exists multiple methods for simplifying logic circuit includes. The most direct way is to use Boolean algebra, i.e., the algebraic simplification method. However, this method is usually based on experience and often becomes a trial-and-error process. Also, there is no easy way to tell whether a simplified expression is in it simplest form.

There are two essential steps for algebraic simplification method. First, the original expression is put into the sum-of-products form by repeated application of **DeMorgan' s theorem**

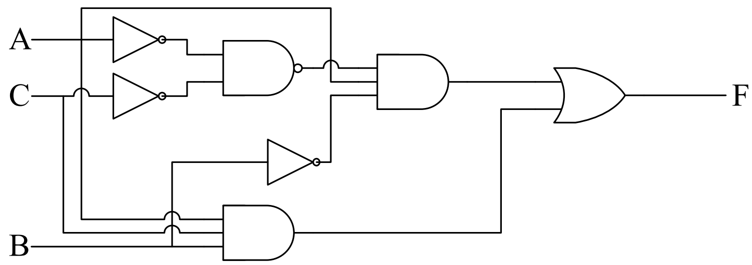


FIGURE 3.1
A complex logic circuit. A typical circuit design before simplification.

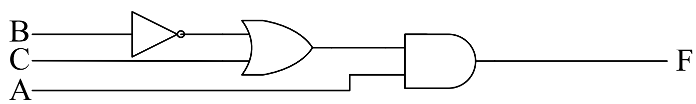


FIGURE 3.2
A simplified logic circuit. The complex circuit in Figure 3.1 after simplification. These 2 circuits are equivalent.

and multiplication of terms. Then the product terms are checked for common factors, and factoring is performed whenever possible.

We also covered some logic. Here is DeMorgan's law:

$$\neg[p \wedge q] \equiv \neg p \vee \neg q, \tag{3.1}$$

and the other form:

$$\neg[p \vee q] \equiv \neg p \wedge \neg q. \tag{3.2}$$

Some examples in Table 3.1 shows that very complex Boolean function can be simplified into a much readable form.

Some special functions that should be remembered is the `XOR` (Exclusive OR) and the `XNOR` (Exclusive NOT OR) gates. The `XOR` gate can be represented as

$$\text{XOR} = A\bar{B} + \bar{A}B = (A + B)(\bar{A} + \bar{B})$$

Original Function	Simplifed function
$ABC + A\bar{B}(\overline{\bar{A}\bar{C}})$	$A(\bar{B} + C)$
$ABC + AB\bar{C} + A\bar{B}C$	$A(B + C)$
$\bar{A}C(\overline{\bar{A}BD}) + \bar{A}B\bar{C}\bar{D} + A\bar{B}C$	$\bar{B}C + \bar{A}\bar{D}(B + C)$
$(\bar{A} + B)(A + B + D)\bar{D}$	$B\bar{D}$

TABLE 3.1
Simplification of Boolean functions. The Boolean functions can be simplified using DeMorgan's law.

and the `XNOR` gate can be represented as

$$\text{XNOR} = (A + \bar{B})(\bar{A} + B) = AB + \bar{A}\bar{B}.$$

3.2 Designing Combinational Logic Circuits

A simple way to design a combinational logic circuit is by using the truth table. The procedure goes as follows:

1. Set up the truth table.
2. Write the `AND` term for each case where the output is a 1.
3. Write the sum-of-products expression for the output.
4. Simplify the output expression.
5. Implement the circuit for the final expression.

For example, consider the following problem. For a 3 digit binary number $b_2b_1b_0$, if the value is greater than or equal 3 in the decimal system, then output an 1. The truth table for this problem is shown in Table 3.2.

Converting the truth table in Table 3.2 results in the following Boolean function

$$F = \bar{A}BC + A\bar{B}\bar{C} + A\bar{B}C + AB\bar{C} + ABC.$$

b_2 (A)	b_1 (B)	b_0 (C)	Output (F)
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

TABLE 3.2
A truth table. This is the truth table for the problem in the text.

After some algebraic computations, we can simplify the above function into

$$F = A + BC.$$

3.3 Karnaugh Map Method

In 1953, Maurice Karnaugh was a telecommunications engineer at Bell Labs. While exploring the new field of digital logic and its application to the design of telephone circuits, he invented a graphical way of visualizing and then simplifying Boolean expressions. This graphical representation, now known as a **Karnaugh map**, or *Kmap*, is named in his honor. A Karnaugh map is a matrix consisting of rows and columns that represent the output values of a Boolean function. The output values placed in each cell are derived from the *minterms* of a Boolean function. A minterm is a product term that contains all of the function' s variables exactly once, either complemented or not complemented.

A function with 2 variables x and y has the minterms $\bar{x}\bar{y}$, $\bar{x}y$, $x\bar{y}$, and xy. For the Boolean function $F(x, y) = xy + x\bar{y}$, the minterms are shown in Figure 3.3. Similarly, a function having three inputs has the minterms shown in Table 3.4.

Minterm	X	Y	F
$\bar{x}\bar{y}$	0	0	1
$\bar{x}y$	0	1	0
$x\bar{y}$	1	0	1
xy	1	1	1

TABLE 3.3
Minterms of a Boolean function. Minterms for a Boolean function with 2 variables.

Minterm	X	Y	Z
$\bar{x}\bar{y}\bar{z}$	0	0	0
$\bar{x}\bar{y}z$	0	0	1
$\bar{x}y\bar{z}$	0	1	0
$\bar{x}yz$	0	1	1
$x\bar{y}\bar{z}$	1	0	0
$x\bar{y}z$	1	0	1
$xy\bar{z}$	1	1	0
xyz	1	1	1

TABLE 3.4
Minterms of a Boolean function. Minterms for a Boolean function with 3 variables.

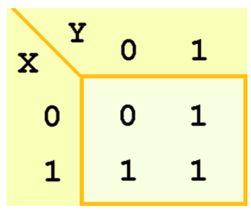
A Karnaugh map has a cell for each minterm, which means that for each line in the truth table, there is a corresponding cell. For the function $F(X, Y) = X + Y$, the truth table is shown in Table 3.5. This function is equivalent to the `OR` of all of the minterms that have a value of 1, thus,

$$F(X, Y) = X + Y = \bar{X}Y + x\bar{Y} + XY.$$

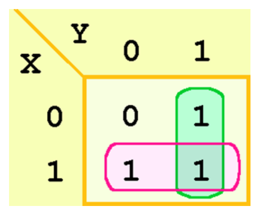
However, this function is not in its simplest form yet. It can be reduced by using the Karnaugh map by finding adjacent **1**s in the Karnaugh map that can be collected into groups that are powers of two. The Karnaugh map for function F is shown in Figure 3.3(a). In this example, we have 2 groups as shown in Figure 3.3(b).

Minterm	X	Y	F
$\bar{x}\bar{y}$	0	0	0
$\bar{x}y$	0	1	1
$x\bar{y}$	1	0	1
xy	1	1	1

TABLE 3.5
Truth table for $F(X,Y) = X + Y$.



(a) The Karnaugh map for function F.



(b) Groups found in the Karnaugh map.

FIGURE 3.3
A 4x4 Karnaugh map. The Karnaugh map can be used to simplify Boolean logic functions.

The rules of Karnaugh map simplification are:

1. Groupings can contain only 1s; no 0s.
2. Groups can be formed only at right angles; diagonal groups are not allowed.
3. The number of 1s in a group must be a power of 2 –even if it contains a single 1.
4. The groups must be made as large as possible.
5. Groups can overlap and wrap around the sides of the Karnaugh map.

A Karnaugh map for three variables is constructed as shown in Figure 3.4. Notice that the values for the YZ combination at the top of the matrix form a pattern that is not a normal binary sequence. This is to allow the variable pair to change value one at a time. Figure 3.4 is read as follows. The first row contains all minterms where X has a value of zero. The first column contains all minterms where Y and Z both have a value of zero. The second column contains all minterms where Y has a value of 0 and Z has a value of 1.

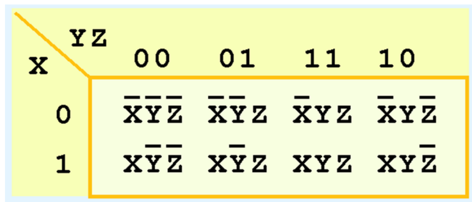


FIGURE 3.4
A 3 variable Karnaugh map. The minterms are listed in each cell.

Consider the function

$$F(X,Y,Z) = \bar{X}\bar{Y}Z + \bar{X}YZ + X\bar{Y}Z + XYZ.$$

The Karnaugh map for $F(X,Y,Z)$ is shown in Figure 3.5. By grouping all the 1s, it shows us that changes in the variables X and Y have no influence upon the value of the function. Finally, this Boolean function can be simplified to $F(X,Y,Z) = Z$.

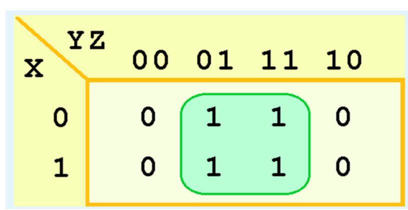


FIGURE 3.5
A 3 variable Karnaugh map. A 3 variable Karnaugh map that can be simplified.

Consider the function

$$F(X,Y,Z) = \bar{X}\bar{Y}\bar{Z} + \bar{X}\bar{Y}Z + \bar{X}YZ + \bar{X}Y\bar{Z} + X\bar{Y}\bar{Z} + XY\bar{Z}.$$

The Karnaugh map for $F(X,Y,Z)$ is shown in Figure 3.6. This example shows a group that wraps around the sides of a Karnaugh map, telling us that the values of X and Y are not relevant to the term of the function that is encompassed by the group. The green group in the top row tells us that only the value of X is significant in that group. It is complemented in that row, so the other term of the reduced function is $\bar{X}$. Finally, this Boolean function can be simplified to $F(X,Y,Z) = \bar{X} + \bar{Z}$.

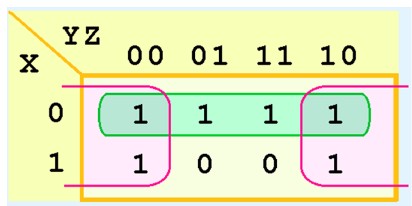


FIGURE 3.6
A 3 variable Karnaugh map. A 3 variable Karnaugh map that can be simplified.

This model can be extended to 4 variables as shown in Figure 3.7. Consider the function

$$F(W,X,Y,Z) = \bar{W}\bar{X}\bar{Y}\bar{Z}+\bar{W}\bar{X}\bar{Y}Z+\bar{W}\bar{X}Y\bar{Z}+\bar{W}XY\bar{Z}+W\bar{X}\bar{Y}\bar{Z}+W\bar{X}\bar{Y}Z+W\bar{X}Y\bar{Z}$$

and is mapping on Figure 3.8. By picking the group cross the edges of the Karnaugh map, we can simplify the function to

$$F(W,X,Y,Z) = \bar{W}\bar{Y} + \bar{X}\bar{Z} + \bar{W}Y\bar{Z}.$$

It is possible to have multiple-choice as to how to pick groups within a Karnaugh even while keeping the groups as large as possible. The (different) functions that result from the groupings below are logically equivalent, i.e., which are all correct solutions.

For further study, there are also terms call "**don't care**" terms which can be used to assist in simplifying the functions. The Karnaugh map for variables greater than 6 is hard to decipher. Other methods for more variables also exist, such as the Quinn-McCluskey method.

WX \ YZ	00	01	11	10
00	$\overline{W}\overline{X}\overline{Y}\overline{Z}$	$\overline{W}\overline{X}\overline{Y}Z$	$\overline{W}\overline{X}YZ$	$\overline{W}\overline{X}Y\overline{Z}$
01	$\overline{W}X\overline{Y}\overline{Z}$	$\overline{W}X\overline{Y}Z$	$\overline{W}XYZ$	$\overline{W}XY\overline{Z}$
11	$WX\overline{Y}\overline{Z}$	$WX\overline{Y}Z$	$WXYZ$	$WXY\overline{Z}$
10	$W\overline{X}\overline{Y}\overline{Z}$	$W\overline{X}\overline{Y}Z$	$W\overline{X}YZ$	$W\overline{X}Y\overline{Z}$

FIGURE 3.7
A 4 variable Karnaugh map. The miniterms are shown in each cell.

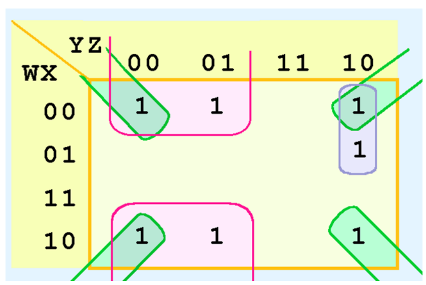


FIGURE 3.8
A 4 variable Karnaugh map. The simplification process can cross the edges of the Karnaugh map.

3.4 Lab Questions

For the questions below, please write down the Boolean functions, truth tables, the simplification method and process, and the final circuit using **LogicCircuit**.

1. (50) Design a logic circuit which implements a majority voter. There are 3 people A, B, and C. If 2 out of the three people votes for an yes, i.e., a logic **1**, then the output is 1. Otherwise, output 0.

2. (50) Design a logic circuit that is to produce a 1 or `HIGH` output when the voltage (represented by a four-bit binary number ABCD) is greater than 6V.

3.5 Lab Report

Your lab report is due in class or no later than 1 week after the lab.